International Journal of Theoretical Physics, Vol. 35, No. 2, 1996

Asymptotic Equivalence of Abstract Impulsive Differential Equations

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Received January 27, 1993

The notion of (h, k)-dichotomy is introduced, which is a generalization of the classical exponential dichotomy. By means of the Schauder–Tychonoff theorem an asymptotic equivalence is proved between a linear impulsive differential equation which is (h, k)-dichotomous and the corresponding perturbed nonlinear equation.

1. INTRODUCTION

The beginning of the development of the theory of abstract impulsive differential equations was marked by the publication of a cycle of papers in the period 1987–1991 (Bainov *et al.*, 1988a–c, 1989a–c, 1990a,b, 1991; Zabreiko *et al.*, 1988).

In the present paper the asymptotic equivalence between a linear impulsive differential equation and its corresponding nonlinear perturbed equation is investigated. This work was influenced by the ideas of Naulin and Pinto (n.d.).

2. STATEMENT OF THE PROBLEM

Let X be an arbitrary Banach space with identity operator I. Denote by $L(X)$ the space of all linear bounded operators acting in X. Consider the impulsive differential equations

$$\frac{dy}{dt} = A(t)y + F(t, y) \qquad (t \neq t_n) \tag{1}$$

$$y(t_n^+) = (Q_n + R_n)y(t_n) \qquad (n = 1, 2, 3, \ldots) \tag{2}$$

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0020-7748/96/0200-0383$09.50/0 © 1996 Plenum Publishing Corporation

and

$$\frac{dx}{dt} = A(t)x \qquad (t \neq t_n) \tag{3}$$

$$x(t_n^+) = Q_n x(t_n) \tag{4}$$

where $t \in \mathbb{R}_+ = [0, \infty)$ and $x(t), y(t) \in X$ ($t \in \mathbb{R}_+$). The solutions x, y at $t = t_n$ are assumed to be continuous from the left.

We shall say that conditions (H) are satisfied if the following conditions hold:

H1. The sequence of points of impulse effect $T = \{t_1, t_2, \ldots\} \subset (0, \infty)$ satisfies the conditions

$$t_n < t_{n+1}, \qquad \lim_{n \to \infty} t_n = \infty$$

H2. The operator-valued function A: $\mathbb{R}_+ \setminus \{t_n\} \to L(X)$ is continuously extendable from (t_n, t_{n+1}) to $[t_n, t_{n+1}]$ ($n = 0, 1, \ldots; t_0 \geq 0$).

H3. The function F: $(\mathbb{R}_+ \setminus \{t_n\}) \times X \to X$ is continuously extendable from $(t_n, t_{n+1}) \times X$ to $[t_n, t_{n+1}] \times X$ ($n = 0, 1, \ldots$).

H4. $Q_n \in L(X)$ and there exist $Q_n^{-1} \in L(X)$.

H5. R_n: $X \to X$ are continuous operators.

Let conditions (H) hold. By $U(t)$ ($0 \leq t < \infty$) denote the evolutionary operator of the linear equation (3), (4) (e.g., Zabreiko *et al.*, 1988; Bainov *et al.*, 1989a,c), i.e., the function $t \mapsto U(t)x_0$ is a solution of equation (3), (4) with initial condition $x(0) = x_0$ ($\forall x_0 \in X$).

Definition 1. Let h, k: $\mathbb{R}_+ \to (0, \infty)$ be two functions such that h^{-1} and k^{-1} are continuously extendable from (t_n, t_{n+1}) to $[t_n, t_{n+1}]$ and let P be a projector acting in X. The linear impulsive equation (3), (4) is said to be *(h, k)-dichotomous* if there exists a constant $K > 0$ such that

$$\|U(t)PU^{-1}(s)\| \leq Kh(t)h^{-1}(s), \qquad 0 \leq s \leq t \tag{5}$$

$$\|U(t)(I - P)U^{-1}(s)\| \leq Kh^{-1}(t)k(s), \qquad 0 \leq t \leq s \tag{6}$$

$h^{-1} = 1/h$ and $k^{-1} = 1/k$.

Remark 1. We note that for $h(\tau) = k(\tau) = e^{-\delta\tau}$ we obtain the exponential dichotomy of the impulsive equation (3), (4) (e.g., Bainov *et al.*, 1989c, 1991). If the functions $h(\tau)$ and $k(\tau)$ are differentiable, we obtain the (μ_1, μ_2)-dichotomy from Muldowney (1984).

Let $0 \leq t_0$. By $C(t_0)$ denote the set of all functions f: $[t_0, \infty) \to X$ which are continuous for $t \in [t_0, \infty) \setminus \{t_n\}$, have discontinuities of the first kind at

$t = t_n \geq t_0$, and are continuous from the left. With respect to the family of seminorms ($t_0 < \Delta < \infty$)

$$P_\Delta(f) = \sup_{t_0 \leq t \leq \Delta} \|f(t)\|$$

the set $C(t_0)$ is a locally convex, metrizable space.

3. MAIN RESULTS

By B_r we shall denote the closed ball in X of center O and radius r. Set

$$D_r = \{x(\cdot) \in C(t_0): h^{-1}(t)x(t) \in B_r (t \geq t_0)\} \tag{7}$$

Let $x \in C(t_0)$. Consider the operator

$$(Qy)(t) = x(t) + \int_{t_0}^{\infty} G(t, s)F(s, y(s))\, ds + \sum_{j:\, t_0 \leq t_j} G(t, t_j)H_j(y(t_j)) \qquad (t \geq t_0) \tag{8}$$

where

$$G(t, s) = \begin{cases} U(t)PU^{-1}(s), & 0 \leq s \leq t \\ -U(t)(I - P)U^{-1}(s), & 0 \leq t < s \end{cases} \tag{9}$$

is the Green's operator-valued function for the linear equation (3), (4) (Bainov *et al.*, 1989a, 1991) and $H_j = Q_j^{-1}R_j$.

Lemma 1. Let the following conditions hold:

1. Conditions (H) are met.
2. The linear equation (3), (4) is (h, k)-dichotomous.
3. There exist constants $C, C_1 \geq 1$ for which the following inequalities are valid: $h(t)k(t)h^{-1}(s)k^{-1}(s) \leq C$, $h(t)k(t) \leq C_1$ $(0 \leq s \leq t < \infty)$.
4. $\|F(t, y)\| \leq r(t, y)\|y\|$, $t \geq 0$, $y \in X$, where

$$\int_{t_0}^{\infty} \sup_{\|x\| \leq \rho} r(t, h(t)x)\, dt \leq m(\rho, t_0) < \infty$$

5. $\|H_j(x)\| \leq m_j\|x\|$ $(j = 1, 2, \ldots)$, where $\sum_{j=1}^{\infty} m_j \leq M$.
6. For some σ and ρ the inequality $\sigma + K\rho C m(\rho, t_0) + K\rho CM \leq \rho$ is valid.
7. The sets $F([t_0, \infty) \times B_\rho)$ and $\bigcup_{j=1}^{\infty} H_j(B_\rho)$ are relatively compact in X.

Then for each $x \in D_\sigma$ the operator Q has a fixed point in D_ρ.

Proof. In order to prove Lemma 1, we shall show that:
(i) $Q: D_\rho \to D_\rho$.
(ii) Q is a continuous operator.
(iii) QD_ρ is a relatively compact set.

(i) Let $y(\cdot) \in D_\rho$ be an arbitrary element. Then for

$$h^{-1}(t)(Qy)(t)$$
$$= h^{-1}(t)x(t) + \int_{t_0}^{\infty} h^{-1}(t)G(t, s)F(s, y(s))\, ds$$
$$+ \sum_{j:\ t_0 \le t_j} h^{-1}(t)G(t, t_j)H_j(y(t_j)) \tag{10}$$

from relations (9), (5), (6) we deduce the estimate

$$\|h^{-1}(t)(Qy)(t)\|$$
$$\le \|h^{-1}(t)x(t)\| + K\int_{t_0}^{t} h^{-1}(t)h(t)h^{-1}(s)h(s)r(s, y(s))h^{-1}(s)\|y(s)\|\, ds$$
$$+ K\int_{t}^{\infty} h^{-1}(t)k^{-1}(t)k(s)h(s)r(s, y(s))h^{-1}(s)\|y(s)\|\, ds$$
$$+ K\sum_{j:\ t_j \le t} h^{-1}(t)h(t)h^{-1}(t_j)h(t_j)h^{-1}(t_j)m_j\|y(t_j)\|$$
$$+ K\sum_{j:\ t \le t_j} h^{-1}(t)k^{-1}(t)k(t_j)h(t_j)h^{-1}(t_j)m_j\|y(t_j)\| \tag{11}$$

and, consequently,

$$\|h^{-1}(t)(Qy)(t)\|$$
$$\le \sigma + K\rho\int_{t_0}^{t} r(s, y(s))\, ds + K\rho C\int_{t}^{\infty} r(s, y(s))\, ds$$
$$+ K\rho\sum_{t_j \le t} m_j + K\rho C\sum_{t \le t_j} m_j$$
$$\le \sigma + K\rho C\int_{t_0}^{\infty} r(s, y(s))\, ds + K\rho C\sum_{j=1}^{\infty} m_j$$
$$\le \sigma + K\rho C m(\rho, t_0) + K\rho C M$$

Assertion (i) follows from condition 6 of Lemma 1.

(ii) Let $\{y_n\} \subset D_\rho$ be an arbitrary sequence tending to $y_0 \in D_\rho$, i.e., for each $l > t_0$ we have

$$\lim_{n\to\infty} \sup_{t\in[t_0,l]} \|y_n(t) - y_0(t)\| = 0$$

Let $\epsilon > 0$ be an arbitrarily chosen positive number. Choose the number $l_0 \geq l$ so that

$$K\rho C \int_{l_0}^{\infty} r(s, z)\, ds < \frac{\epsilon}{4} \qquad (z \in B_\rho) \tag{12}$$

and

$$\sum_{l_0<t_j} m_j < \frac{\epsilon}{4K\rho C} \tag{13}$$

For $h^{-1}(t)\|Qy_0(t) - Qy_n(t)\|$ $(t_0 \leq t \leq l)$ we obtain the estimate

$$\begin{aligned}
&h^{-1}(t)\|Qy_0(t) - Qy_n(t)\| \\
&\quad \leq K \int_{t_0}^{t} h^{-1}(t)h(t)h^{-1}(s)\|F(s, y_0(s)) - F(s, y_n(s))\|\, ds \\
&\qquad + K \int_{t}^{\infty} h^{-1}(t)k^{-1}(t)k(s)\|F(s, y_0(s)) - F(s, y_n(s))\|\, ds \\
&\qquad + K \sum_{t_j<t} h^{-1}(t)h(t)h^{-1}(t_j)\|H_j(y_0(t_j)) - H_j(y_n(t_j))\| \\
&\qquad + K \sum_{t\leq t_j} h^{-1}(t)k^{-1}(t)k(t_j)\|H_j(y_0(t_j)) - H_j(y_n(t_j))\| \\
&\quad \leq K \int_{t_0}^{t} h^{-1}(s)\|F(s, y_0(s)) - F(s, y_n(s))\|\, ds \\
&\qquad + K \int_{t_0}^{l_0} h^{-1}(t)k^{-1}(t)k(s)\|F(s, y_0(s)) - F(s, y_n(s))\|\, ds \\
&\qquad + K \int_{l_0}^{\infty} h^{-1}(t)k^{-1}(t)k(s)\|F(s, y_0(s)) - F(s, y_n(s))\|\, ds \\
&\qquad + K \sum_{t_j<t} h^{-1}(t_j)\|H_j(y_0(t_j)) - H_j(y_n(t_j))\| \\
&\qquad + K \sum_{t_0\leq t_j\leq l_0} h^{-1}(t)k^{-1}(t)k(t_j)\|H_j(y_0(t_j)) - H_j(y_n(t_j))\| \\
&\qquad + K \sum_{l_0<t_j} h^{-1}(t)k^{-1}(t)k(t_j)\|H_j(y_0(t_j)) - H_j(y_n(t_j))\|
\end{aligned} \tag{14}$$

The first two integrals and the first two sums on the right-hand side of the last inequality tend to zero as $n \to \infty$ uniformly with respect to $t \in [t_0, l]$. The value of the third integral does not exceed

$$
\begin{aligned}
&K \int_{l_0}^{\infty} h^{-1}(t)k^{-1}(t)k(s)\|F(s, y_0(s))\| \, ds \\
&\quad + K \int_{l_0}^{\infty} h^{-1}(t)k^{-1}(t)k(s)\|F(s, y_n(s))\| \, ds \\
&\quad \leq K\rho C \int_{l_0}^{\infty} r(s, y_0(s)) \, ds + K\rho C \int_{l_0}^{\infty} r(s, y_n(s)) \, ds < \frac{\epsilon}{2}
\end{aligned}
$$

For the third sum we obtain the estimate

$$
\begin{aligned}
&K \sum_{l_0<t_j} h^{-1}(t)k^{-1}(t)k(t_j)\|H_j(y_0(t_j))\| \\
&\quad + K \sum_{l_0<t_j} h^{-1}(t)k^{-1}(t)k(t_j)\|H_j(y_n(t_j))\| \\
&\quad \leq KC\rho \sum_{l_0<t_j} m_j + KC\rho \sum_{l_0<t_j} m_j < \frac{\epsilon}{2}
\end{aligned}
$$

Thus assertion (ii) is proved.

(iii) It suffices to prove the equicontinuity of the functions of $M\rho = QD_\rho$ ($QD_\rho \subset D\rho$) on each interval $(t_n, t_{n+1}) \subset (t_0, \infty)$ (for the first one instead of t_n we must take t_0). Let $s, t \in (t_n, t_{n+1})$ and let $s \leq t \leq s + \delta$. We shall prove that for sufficiently small values of $\delta > 0$ the following inequality is valid:

$$
\|(Qy)(t) - (Qy)(s)\| < \epsilon \qquad (y \in D\rho) \tag{15}
$$

From (8) for $(Qy)(t) - (Qy)(s)$ we obtain the representation

$$
\begin{aligned}
&(Qy)(t) - (Qy)(s) \\
&\quad = x(t) - x(s) + \int_{t_0}^{t} U(t)PU^{-1}(u)F(u, y(u)) \, du \\
&\qquad - \int_{t}^{\infty} U(t)(I - P)U^{-1}(u)F(u, y(u)) \, du \\
&\qquad - \int_{t_0}^{s} U(s)PU^{-1}(u)F(u, y(u)) \, du + \int_{s}^{\infty} U(s)(I - P)U^{-1}(u)F(u, y(u)) \, du
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{t_0 \le t_j < t} U(t)PU^{-1}(t_j)H_j(y(t_j)) - \sum_{t \le t_j} U(t)(I - P)U^{-1}(t_j)H_j(y(t_j)) \\
&- \sum_{t_0 \le t_j < s} U(s)PU^{-1}(t_j)H_j(y(t_j)) \\
&+ \sum_{s \le t_j} U(s)(I - P)U^{-1}(t_j)H_j(y(t_j)) \\
= {}& x(t) - x(s) + \int_{t_0}^{s} (U(t) - U(s))PU^{-1}(u)F(u, y(u))\, du \\
&- \int_{s}^{\infty} (U(t) - U(s))(I - P)U^{-1}(u)F(u, y(u))\, du \\
&+ \int_{s}^{t} U(t)PU^{-1}(u)F(u, y(u))\, du \\
&+ \int_{s}^{t} U(t)(I - P)U^{-1}(u)F(u, y(u))\, du \\
&+ \sum_{t_0 \le t_j < s} (U(t) - U(s))PU^{-1}(t_j)H_j(y(t_j)) \\
&- \sum_{s \le t_j} (U(t) - U(s))(I - P)U^{-1}(t_j)H_j(y(t_j)) \\
= {}& x(t) - x(s) + \int_{t_0}^{s} (U(t)U^{-1}(s) - I)U(s)PU^{-1}(u)F(u, y(u))\, du \\
&- \int_{s}^{\infty} (U(t)U^{-1}(s) - I)U(s)(I - P)U^{-1}(u)F(u, y(u))\, du \\
&+ \int_{s}^{t} U(t)U^{-1}(u)F(u, y(u))\, du \\
&+ \sum_{t_0 \le t_j < s} (U(t)U^{-1}(s) - I)U(s)PU^{-1}(t_j)H_j(y(t_j)) \\
&- \sum_{s \le t_j} (U(t)U^{-1}(s) - I)U(s)(I - P)U^{-1}(t_j)H_j(y(t_j))
\end{aligned} \tag{16}
$$

Choose δ so small that $\|x(t) - x(s)\| < \epsilon/6$ and $\|I - U(t)U^{-1}(s)\| < \min\{a, b, c, d\}$, where

$$
a = \frac{\epsilon}{6} \frac{1}{K\rho h(s) m(\rho, t_0)}
$$

$$
b = \frac{\epsilon}{6} \frac{k(s)}{K\rho C_1 m(\rho, t_0)}
$$

$$c = \frac{\epsilon}{6} \frac{1}{K\rho C h(s) M}$$

$$d = \frac{\epsilon}{6} \frac{k(s)}{K\rho C_1 M}$$

Then

$$\begin{aligned}
&\|Qy(t) - Qy(s)\| \\
&\quad < \frac{\epsilon}{6} + aK \int_{t_0}^{s} h(s)h^{-1}(u)h(u)r(u, y(u))\|y(u)\|h^{-1}(u)\, du \\
&\qquad + bK \int_{s}^{\infty} k^{-1}(s)k(u)h(u)r(u, y(u))\|y(u)\|h^{-1}(u)\, du \\
&\qquad + \int_{s}^{t} \|U(t)U^{-1}(u)\| r(u, y(u))\|y(u)\|\, du \\
&\qquad + cK \sum_{t_0 \le t_j < s} h(s)h^{-1}(t_j)m_j\|y(t_j)\| \\
&\qquad + dK \sum_{s \le t_j} k^{-1}(s)k(t_j)m_j\|y(t_j)\|
\end{aligned}$$

Choose δ so small that the following inequality should hold, too:

$$\rho \int_{s}^{t} \|U(t)U^{-1}(u)\| h(u)r(u, y(u))\, du < \frac{\epsilon}{6}$$

Then

$$\begin{aligned}
&\|Qy(t) - Qy(s)\| \\
&\quad < \frac{\epsilon}{6} + aK\rho h(s) \int_{t_0}^{s} r(u, y(u))\, du + bK\rho k^{-1}(s) \\
&\qquad \times \int_{s}^{\infty} k(u)h(u)r(u, y(u))\, du + \frac{\epsilon}{6} + cK\rho h(s) \sum_{t_0 \le t_j < s} m_j \\
&\qquad + dK\rho k^{-1}(s) \sum_{s \le t_j} k(t_j)h(t_j)m_j \\
&\quad \le \frac{\epsilon}{6} + aK\rho h(s)m(\rho, t_0) + bK\rho k^{-1}(s)C_1 m(\rho, t_0)
\end{aligned}$$

$$+ \frac{\epsilon}{6} + cK\rho h(s)M + dK\rho k^{-1}(s)C_1M = \epsilon$$

From condition 7 of Lemma 1 there follows the compactness in X of the sets

$$H_t = \{(Qz)(t): z \in D\rho\} \qquad (t \geq t_0) \quad \blacksquare$$

Remark 1. For dim $X < \infty$ the assertion of Lemma 1 is still valid without condition 7.

Theorem 1. Let the conditions of Lemma 1 hold, where the function x is a solution of the linear impulsive equation (3), (4).

Then the fixed point $y(\cdot)$ of the operator Q is a solution of the nonlinear equation (1), (2).

Proof. Taking into account that x is a solution of (3), (4) and the equality

$$W_1(t, t) + W_2(t, t) = I$$

where $W_1(t, s) = U(t)PU^{-1}(s)$ and $W_2(t, s) = U(t)(I - P)U^{-1}(s)$, for $t \notin \{t_n\}$ we obtain

$$\begin{aligned} y'(t) &= (Qy)'(t) \\ &= x'(t) + W_1(t, t)F(t, y(t)) \\ &\quad + A(t)\int_{t_0}^{t} W_1(t, s)F(s, y(s))\,ds + W_2(t, t)F(t, y(t)) \\ &\quad - A(t)\int_{t}^{\infty} W_2(t, s)F(s, y(s))\,ds + A(t)\sum_{t_0 \leq t_j < t} W_1(t, t_j)H_j(y(t_j)) \\ &\quad - A(t)\sum_{t < t_j < \infty} W_2(t, t_j)H_j(y(t_j)) \\ &= A(t)y(t) + F(t, y(t)) \end{aligned}$$

Let $t = t_n$. Then

$$\begin{aligned} y(t_n^+) &= (Qy)(t_n^+) \\ &= Q_nx(t_n) + \int_{t_0}^{t_n} Q_nW_1(t_n, s)F(s, y(s))\,ds \\ &\quad - \int_{t_n}^{\infty} Q_nW_2(t_n, s)F(s, y(s))\,ds + \sum_{t_0 \leq t_j \leq t_n} Q_nW_1(t_n, t_j)H_j(y(t_j)) \end{aligned}$$

$$- \sum_{j=n+1}^{\infty} Q_n W_2(t_n, t_j) H_j(y(t_j))$$
$$= Q_n[y(t_n) + W_1(t_n, t_n) H_n(y(t_n))$$
$$+ W_2(t_n, t_n) H_n(y(t_n))]$$
$$= Q_n y(t_n) + Q_n H_n(y(t_n))$$
$$= Q_n y(t_n) + R_n(y(t_n)) \quad \blacksquare$$

Theorem 2. Let the conditions of Lemma 1 hold.

Then for any solution $y \in D_\sigma$ of the nonlinear impulsive equation (1), (2) there exists a solution $x \in D_\rho$ of (3), (4).

Proof. Consider the function

$$\begin{aligned} x(t) = y(t) &- \int_{t_0}^{t} U(t) P U^{-1}(s) F(s, y(s))\, ds \\ &+ \int_{t}^{\infty} U(t)(I - P) U^{-1}(s) F(s, y(s))\, ds \\ &- \sum_{t_0 < t_j < t} U(t) P U^{-1}(t_j) H_j(y(t_j)) \\ &+ \sum_{t \le t_j} U(t)(I - P) U^{-1}(t_j) H_j(y(t_j)) \end{aligned} \tag{17}$$

It is not hard to check that the function $x(t)$ is correctly defined and satisfies equation (3), (4). ■

Theorem 3. Let the following conditions hold:

1. The conditions of Lemma 1 are met.
2. The following condition holds:

$$\begin{aligned} &\int_{t_0}^{t} h(t) h^{-1}(s) \sup_{\|x\| \le \rho} r(s, h(s))\, ds \\ &+ \int_{t}^{\infty} k^{-1}(s) k(t) \sup_{\|x\| \le \rho} r(s, h(s))\, ds \\ &+ \sum_{t_j < t} h(t) h^{-1}(t_j) m_j + \sum_{t \le t_j} k^{-1}(t_j) k(t) m_j \to 0 \qquad (t \to \infty) \end{aligned}$$

3. $KC \lim_{t_0 \to \infty} \sup_{\rho > 0} m(\rho, t_0) + KCM < 1$.

Then equations (1), (2) and (3), (4) are asymptotically equivalent, i.e., for any bounded solution $y(\cdot)$ of (1), (2) there exists a bounded solution $x(\cdot)$ of (3), (4) and, conversely, for any bounded solution $x(\cdot)$ of (3), (4) there exists a bounded solution $y(\cdot)$ of (1), (2) such that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0$$

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